JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 21, No. 6, November–December 1998

Nonlinear H_∞ Robust Guidance Law for Homing Missiles

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This paper proposes an H_∞ robust guidance law for homing missiles with nonlinear kinematics in the homing phase. Unlike conventional approaches where target's acceleration is often assumed to be known or needs to be estimated in real time, the proposed robust guidance law can achieve performance robustness in the absence of target's acceleration information and under variations of the initial conditions of engagement. The most difficult and challenging task involved in applying nonlinear H_∞ control theory is the solution of the associated Hamilton–Jacobi partial differential inequality. In this paper we show that the Hamilton–Jacobi partial differential inequality of the missile guidance problem can be solved analytically with simple manipulations. The numerical simulations show that the H_∞ robust guidance law exhibits strong robustness properties against disturbances from target's maneuvers and variations in initial engagement conditions.

I. Introduction

A SYSTEM in the presence of unknown parameters or disturbances often has poor performance or even becomes unstable when the controller is optimized with respect to a nominal model of the system or with respect to a fixed disturbance. Hence, robustness of performance with respect to model variations and/or disturbance variations must be considered in control system design. For homing missile guidance systems, the model uncertainties and disturbances include varying parameters within missile dynamics, the variations of the target's maneuvers, and the changing engagement conditions.

To handle system uncertainties and exogenous disturbances, many robust control techniques have been proposed in recent years, for example, H_∞ control, μ-synthesis control, etc. Their applications to aerospace engineering include H_∞ flight control,[1,2] missile autopilot design,[3,4] aeroassisted orbital transfer,[5] and spacecraft attitude control.[6] As to the robustness in missile guidance law design, the guidance law considered in Ref. 7 took into account the uncertainty of the time constant τ within the guidance loop. However, systematic application of modern H_∞ robust control philosophy to guidance law design is still very rare in the literature. The present paper intends to introduce the H_∞ robust control concept to the guidance law design for homing missiles.

Most of the applications of H_∞ control theory to aerospace engineering are limited to linearized systems. The kinematics involved in the missile interception scenario are completely nonlinear, and the linearized kinematics can only be valid near the pursuit end. Therefore, we need a nonlinear version of the H_∞ control theory to tackle the missile interception problem. Although nonlinear H_∞ control theory has been developed successfully,[8–10] its practical application is still questionable due to the difficulties in solving the associated Hamilton–Jacobi partial differential inequality (HJPDI). Constant endeavor has been made in the literature to investigate the solvability of HJPDI. Algebraic and geometric tools[6,11] were used to find a particular solution of the HJPDI for the satellite attitude control problem. Nevertheless, a numerical solution is believed to be a more systematic way to find the solutions of HJPDI. Numerical algorithms[12,13] were developed for finding the Taylor series solutions of HJPDI.

The present paper applies the nonlinear H_∞ control theory to guidance law design, which, as far as the authors can determine, has not been considered in the literature before. Under this approach, the exact nonlinear governing equations are considered. The guidance problem is formulated as a nonlinear disturbance attenuation H_∞ control problem where the target's accelerations are regarded as unpredictable disturbances, and the purpose of the control design is to

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Received Nov. 26, 1997; revision received March 31, 1998; accepted for publication March 31, 1998.

attenuate the influence of the target's maneuvers on the performance of the guidance law, such as miss distance and energy consumption, etc. The characterization of the nonlinear H_∞ guidance law relies on the solution of HJPDI, which is a second-order nonlinear partial differential inequality. At first glance, this HJPDI is rather complicated and hardly to be solved. However, after detailed investigation, a nice analytical solution is found to be possible via simple manipulations.

The robust H_∞ guidance law obtained from the solution of HJPDI exhibits strong performance robustness properties against target maneuvers and against variations of initial engagement conditions. Comparisons with conventional proportional navigation (PN)[14–17] schemes show that the conventional PN schemes do perform very well in the absence of the target's maneuvers. However, when target maneuvers are present, their performances may degrade dramatically. On the other hand, the derived H_∞ robust guidance law may not perform better than the PN schemes for nonmaneuvering targets, but it demonstrates strong performance robustness under the variation of target maneuvers.

Augmented proportional navigation[18] (APN) is a modified PN taking into account target maneuvers. The way that the H_∞ guidance law handles maneuvering targets is basically different from that of APN. For the H_∞ guidance law, we assume that the target's acceleration is unpredictable, whereas for APN the target's acceleration is recursively estimated from a target model. Therefore, the situation is something like the comparison between robust control (H_∞) and adaptive control (APN). We cannot say which is better because the answer depends on the circumstances to which they are applied. In the circumstance where target's acceleration can be soundly estimated, APN is undoubtedly superior to the H_∞ guidance law, whereas in the circumstance where the target's acceleration is unknown or is poorly estimated, the H_∞ guidance law could be better than APN.

This paper is organized as follows. In Sec. II, we briefly survey some preliminaries of the nonlinear H_∞ control theory. In Sec. III, we formulate the missile guidance problem as a nonlinear disturbance attenuation H_∞ control problem. Then we solve analytically the associated HJPDI in Sec. IV. Performance of the H_∞ guidance laws from the HJPDI's solutions is evaluated in Sec. V. Finally, comparisons with PN schemes and the robustness of the H_∞ guidance laws against target maneuvers are illustrated numerically in Sec. VI.

II. Nonlinear H_∞ Control Theory

In this section, we introduce the standard results of the nonlinear H_∞ control theory for later use. Consider a nonlinear state-space system:

$$\dot{\boldsymbol{x}} = f(x) + g(x)\boldsymbol{w}, \qquad \boldsymbol{w} \in R^p, \qquad f(0) = 0 \tag{1a}$$

$$z = h(x), \qquad z \in R^q, \qquad h(0) = 0 \tag{1b}$$

where $\boldsymbol{x}$ is the state vector, and $\boldsymbol{w}$ and z are the exogenous disturbances to be rejected and the penalized output signal, respectively. We assume that $f(x)$, $g(x)$, and $h(x)$ are C^∞ functions and $x = 0$ is the equilibrium point of the system; i.e., $f(0) = h(0) = 0$.

If there exists a scalar C^1 function $U : R^n \longrightarrow R^+$ with $U(0) = 0$ such that

$$\frac{1}{2\gamma^2}\left(\frac{\partial U}{\partial \boldsymbol{x}}\right) g(x)g(x)^T \left(\frac{\partial U}{\partial \boldsymbol{x}}\right)^T + \left(\frac{\partial U}{\partial \boldsymbol{x}}\right) f(x) + \frac{1}{2} h^T(x)h(x) \le 0 \tag{2}$$

then the system is said to have L_2 gain $\le \gamma$; i.e.,

$$\int_0^\infty z^T z \,\mathrm{d}t \le \gamma^2 \int_0^\infty \boldsymbol{w}^T \boldsymbol{w}\,\mathrm{d}t \tag{3}$$

where $(\partial U/\partial x) = [(\partial U/\partial x_1)(\partial U/\partial x_2)\cdots(\partial U/\partial x_n)]$. The details of proof can be found, for example, in Ref. 9.

When control $\boldsymbol{u}$ is applied to the system, we get the controlled system as

$$\dot{\boldsymbol{x}} = f(x) + g_1(x)\boldsymbol{w} + g_2(x)\boldsymbol{u} \tag{4a}$$

$$z = h_1(x) + k_{12}(x)\boldsymbol{u} \tag{4b}$$

where $\boldsymbol{x} \in R^n$, $\boldsymbol{w} \in R^{p1}$, $\boldsymbol{u} \in R^{p2}$, and $z \in R^q$. The terms $f(x)$, $g_1(x)$, $g_2(x)$, $h_1(x)$, and $k_{12}(x)$ are C^∞ functions, and we assume that $h_1^T k_{12} = 0$ and $k_{12}^T k_{12} = I$. These assumptions are introduced to simplify the following derivation, but as shown later these assumptions are satisfied automatically for the missile guidance problem. The nonlinear H_∞ control problem is to find the control $\boldsymbol{u}$ such that the L_2 gain of the closed-loop system is less than or equal to γ. By replacing $f(x)$, $g(x)$, and $h(x)$ in Eq. (2) with $f(x) + g_2(x)u$, $g_1(x)$, and $h_1(x) + k_{12}(x)u$, respectively, the condition that the L_2 gain of the closed-loop system is equal to or lower than γ becomes

$$\frac{1}{2}\left(\boldsymbol{u}^T + \frac{\partial U}{\partial \boldsymbol{x}} g_2\right)\left(\boldsymbol{u}^T + \frac{\partial U}{\partial \boldsymbol{x}} g_2\right)^T + \frac{1}{2}\left(\frac{\partial U}{\partial \boldsymbol{x}}\right) \times \left(\frac{1}{\gamma^2} g_1 g_1^T - g_2 g_2^T\right)\left(\frac{\partial U}{\partial \boldsymbol{x}}\right)^T + \left(\frac{\partial U}{\partial \boldsymbol{x}}\right) f + \frac{1}{2} h_1^T h_1 \le 0 \tag{5}$$

The control $\boldsymbol{u}$ minimizing the left-hand side of the preceding inequality can be easily found as

$$\boldsymbol{u}(x) = -g_2^T(x)\left(\frac{\partial U}{\partial \boldsymbol{x}}\right)^T \tag{6}$$

Substituting this control into Eq. (5), we obtain the desired HJPDI as

$$\left(\frac{\partial U}{\partial \boldsymbol{x}}\right) f + \frac{1}{2}\left(\frac{\partial U}{\partial \boldsymbol{x}}\right)\left(\frac{1}{\gamma^2} g_1 g_1^T - g_2 g_2^T\right)\left(\frac{\partial U}{\partial \boldsymbol{x}}\right)^T + \frac{1}{2} h_1^T h_1 \le 0 \tag{7}$$

Hence, solving the nonlinear H_∞ control problem is equivalent to finding a positive function $U(x)$ satisfying HJPDI. The corresponding HJPDI for the missile guidance problem will be derived in the next section.

The H_∞ guidance law given by Eq. (6) guarantees that the controlled system (4a) is internally stable; i.e., the system can return to the equilibrium point from an initial perturbation x_0 in the absence of external disturbance w. This can be verified by showing that if $U > 0$ is a solution of the HJPDI, then U is a qualified Lyapunov function of the system (4a). To behave as a Lyapunov function, U must satisfy the condition $\dot{U} \le 0$. Substituting Eq. (6) into Eq. (4a) yields $\dot{x} = f(x) - g_2(x)g_2^T(x)(\partial U/\partial x)^T$, which can be used to evaluate $\dot{U}$ as

$$\dot{U} = \left(\frac{\partial U}{\partial \boldsymbol{x}}\right)\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \left(\frac{\partial U}{\partial \boldsymbol{x}}\right) f(x) - \left(\frac{\partial U}{\partial \boldsymbol{x}}\right) g_2(x) g_2^T(x)\left(\frac{\partial U}{\partial \boldsymbol{x}}\right)^T$$

$$\le -\frac{1}{2}\left(\frac{\partial U}{\partial \boldsymbol{x}}\right)\left(\frac{1}{\gamma^2} g_1 g_1^T + g_2 g_2^T\right)\left(\frac{\partial U}{\partial \boldsymbol{x}}\right)^T - \frac{1}{2} h_1 h_1^T \le 0$$

The last inequality is from the HJPDI (7). The properties of $U > 0$ and $\dot{U} \le 0$ show that U is a qualified Lyapunov function, and hence the controlled system (4a) is stable in the sense of Lyapunov. In summary, the nonlinear H_∞ control theory[8–10] guarantees 1) state boundedness (Lyapunov stability) and 2) performance boundedness: $\|z\|_2 \le \gamma\|w\|_2$.

III. Formulation of Robust Guidance Problems

For a planar intercept, relative motion between missile and target is described by the polar coordinates system (r, θ) with the origin fixed on the location of the missile (refer to Fig. 1). For the purpose of guidance law design, missile and target are assumed to be point masses, and only kinematics are considered. The governing equations of the relative motion can be derived as

$$\ddot{r} - r\dot{\theta}^2 = w_r - u_r \tag{8a}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = w_\theta - u_\theta \tag{8b}$$

where r is the relative distance between missile and target; θ is the aspect angle of the line of sight (LOS) with respect to an inertial reference line; w_r and w_θ are the target's acceleration components,

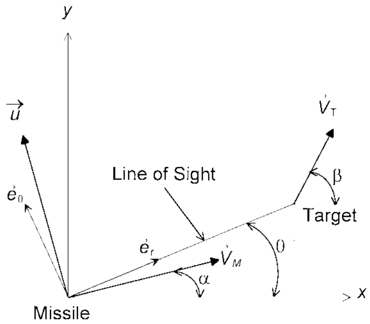

Fig. 1 Planar pursuit geometry.

which are assumed to be an unpredictable disturbance; and u_r and u_θ are the missile's acceleration components, which are to be found. By introducing the new state variables $(r,\ V_r,\ V_\theta)$ where $V_r = \dot{r}$ is the radial relative velocity and $V_\theta = r\dot{\theta}$ is the tangential relative velocity, we can transform Eqs. (8) into the standard state-space form as in Eq. (4a)

$$\begin{bmatrix} \dot{r} \\ \dot{V}_r \\ \dot{V}_\theta \end{bmatrix} = \begin{bmatrix} V_r \\ V_\theta^2/r \\ -V_r V_\theta/r \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} w + \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} u \tag{9}$$

where $\boldsymbol{w} = [w_r \ \ w_\theta]^T$ and $\boldsymbol{u} = [u_r \ \ u_\theta]^T$ are the acceleration vectors of target and missile, respectively.

Next, we need to specify the output signal z to be controlled. A good guidance law must guarantee a decreasing relative distance and at the same time keep the LOS angular rate as small as possible. To reflect this fact, we choose z as

$$z = \begin{bmatrix} \rho h(r,\ V_\theta) \\ \boldsymbol{u} \end{bmatrix} = \begin{bmatrix} \rho h(r,\ V_\theta) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} \boldsymbol{u} \tag{10}$$

where

$$h(r,\ V_\theta) = V_\theta^2/r = r\dot{\theta}^2 \tag{11}$$

is a measure of guidance performance, and ρ is a weighting factor concerning the tradeoff between performance and acceleration command. From the definition of h, it can be seen that when h can be kept small, it implies that the missile is close to the target (small r) and/or the LOS angular rate $\dot{\theta}$ is small. By choosing weighting factor ρ properly, it is possible to obtain an acceptably small h without consuming a great amount of acceleration $\boldsymbol{u}$.

The problem of H_∞ guidance law design now can be stated as follows: find the missile acceleration command $u = [u_r \ \ u_\theta]^T$ such that the L_2 gain of the system described by Eqs. (9) and (10) is lower than or equal to γ; i.e.,

$$\frac{\int_0^\infty z^T z \,\mathrm{d}t}{\int_0^\infty \boldsymbol{w}^T \boldsymbol{w} \,\mathrm{d}t} = \frac{\int_0^\infty (\rho^2 h^2 + \boldsymbol{u}^T \boldsymbol{u}) \,\mathrm{d}t}{\int_0^\infty \left(w_r^2 + w_\theta^2\right) \mathrm{d}t} \le \gamma^2 \qquad \forall \boldsymbol{w} \in \boldsymbol{L}_2 \tag{12}$$

In the preceding equation,

$$\int_0^\infty \boldsymbol{w}^T \boldsymbol{w} \,\mathrm{d}t$$

is the input energy of the system, i.e., the L_2 norm of the disturbance (target's acceleration), whereas

$$\int_0^\infty z^T z \,\mathrm{d}t = \int_0^\infty (\rho^2 h^2 + \boldsymbol{u}^T \boldsymbol{u}) \,\mathrm{d}t = \int_0^\infty \left(\rho^2 r^2 \dot{\theta}^4 + u_r^2 + u_\theta^2\right) \mathrm{d}t$$

denotes the output energy of the system, i.e., the L_2 norm of the desired performance. We wish the output energy to be as small as possible under the action of the disturbance input $\boldsymbol{w}$. The system gain

$$\frac{\int_0^\infty z^T z \,\mathrm{d}t}{\int_0^\infty \boldsymbol{w}^T \boldsymbol{w} \,\mathrm{d}t}$$

can thus be regarded as the disturbance attenuation level, and the H_∞ robust guidance law can guarantee that the disturbance attenuation level is lower than or equal to a specified value γ for the target's acceleration $\boldsymbol{w} = [w_r \ \ w_\theta]^T \in \boldsymbol{L}_2$. It is attributed to the aforementioned inherent property of guaranteed disturbance attenuation level that the H_∞ guidance law can exhibit performance robustness against the variations of target's maneuvers.

IV. Solutions of the Hamilton–Jacobi Partial Differential Inequality

In this section the solutions of the associated HJPDI for the missile guidance problem will be derived analytically. The HJPDI to be solved is established first. Comparing Eqs. (9) and (10) with Eqs. (4), we have

$$x = \begin{bmatrix} r \\ V_r \\ V_\theta \end{bmatrix}, \qquad f = \begin{bmatrix} V_r \\ V_\theta^2/r \\ -V_r V_\theta/r \end{bmatrix}$$

$$g_1 = -g_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad h_1 = \begin{bmatrix} \rho h \\ 0 \end{bmatrix}, \qquad k_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{13}$$

It is noticed that the aforementioned assumptions $h_1^T k_{12} = 0$ and $k_{12}^T k_{12} = I$ are satisfied automatically. Substituting the preceding functions into Eq. (7), we obtain the guidance's HJPDI as

$$V_r\left(\frac{\partial U}{\partial r}\right) + \frac{V_\theta^2}{r}\left(\frac{\partial U}{\partial V_r}\right) - \frac{V_r V_\theta}{r}\left(\frac{\partial U}{\partial V_\theta}\right) + \frac{1}{2}\left(\frac{1}{\gamma^2} - 1\right) \times \left[\left(\frac{\partial U}{\partial V_r}\right)^2 + \left(\frac{\partial U}{\partial V_\theta}\right)^2\right] + \frac{\rho^2 V_\theta^4}{2r^2} \le 0 \tag{14}$$

This is a nonlinear second-order partial differential inequality in the unknown function $U(r,\ V_r,\ V_\theta)$. If a qualified U can be found, the nonlinear H_∞ guidance law is then given by Eq. (6) as

$$\boldsymbol{u}(r,\ V_r,\ V_\theta) = -g_2^T\left(\frac{\partial U}{\partial \boldsymbol{x}}\right)^T = -\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \dfrac{\partial U}{\partial r} & \dfrac{\partial U}{\partial V_r} & \dfrac{\partial U}{\partial V_\theta} \end{bmatrix}^T = \begin{bmatrix} \dfrac{\partial U}{\partial V_r} \\ \dfrac{\partial U}{\partial V_\theta} \end{bmatrix} \tag{15}$$

i.e., $u_r = \partial U/\partial V_r$, and $u_\theta = \partial U/\partial V_\theta$. Hence, the main problem of H_∞ guidance law design is to find a positive function U satisfying the HJPDI in Eq. (14). It can be seen that the nature of the solution depends on the magnitude of the disturbance attenuation level γ: 1) $\gamma > 1$, the fourth term in the HJPDI is always negative; 2) $\gamma = 1$, the HJPDI degenerates into a first-order linear partial differential inequality; and 3) $\gamma < 1$, the fourth term in the HJPDI is always positive. These three kinds of solutions will be derived separately in the following.

A. $\gamma > 1$

In this case, the solution of the HJPDI may be nonunique, but a possible candidate can be found from the relations

$$V_r\left(\frac{\partial U}{\partial r}\right) - \frac{V_r V_\theta}{r}\left(\frac{\partial U}{\partial V_\theta}\right) + \frac{1}{2}\left(\frac{1}{\gamma^2} - 1\right)\left[\left(\frac{\partial U}{\partial V_r}\right)^2 + \left(\frac{\partial U}{\partial V_\theta}\right)^2\right] \le 0 \tag{16a}$$

$$\frac{V_\theta^2}{r}\left(\frac{\partial U}{\partial V_r}\right) + \frac{\rho^2 V_\theta^4}{2r^2} = 0 \tag{16b}$$

It is noted that a function U satisfying Eq. (16) is also a solution of the HJPDI in Eq. (14), but the inverse is not true. The solution of Eq. (16b) can be found readily as

$$U(r,\ V_r,\ V_\theta) = -\frac{\rho^2 V_r V_\theta^2}{2r} + f(r,\ V_\theta) \tag{17}$$

The simplest positive solution is obtained by taking $f(r,\ V_\theta)$ as a constant c. The magnitude of c is determined to guarantee the positiveness of U. If the closing speed V_r is negative, then the first term in U is always positive, which implies that c can be taken as 0. If we allow V_r to be positive, a large enough c must be chosen. To ensure that this c does exist, we need to show that $-\rho^2 V_r V_\theta^2/2r$ is bounded. The term V_r is bounded because V_r is one of the state $\boldsymbol{x} = [r\ \ V_r\ \ V_\theta]$, which is stable in the sense of Lyapunov as proved in Sec. II. The term $V_\theta^2/r = r\dot{\theta}^2 = h$ is bounded for all bounded $\boldsymbol{w}$ by noting Eq. (12). The boundedness of V_r and V_θ^2/r implies that there exists a finite c such that the U in Eq. (17) is positive. However, it needs to be noticed that because the HJPDI depends on the partial differentiations of U and not on the absolute value of U, the final guidance law is independent of the constant c.

To check whether the U given by Eq. (17) is feasible, we substitute this function into Eq. (16a) and obtain

$$\frac{\rho^2 V_r^2 V_\theta^2}{2r^2}\left[3 + \rho^2\left(\frac{1}{\gamma^2} - 1\right)\right] + \left(\frac{1}{\gamma^2} - 1\right)\frac{\rho^4 V_\theta^4}{8r^2} \le 0 \tag{18}$$

Because $\gamma > 1$, the last term of the preceding equation is always negative. The condition that Eq. (18) is satisfied for arbitrary r, V_r, and V_θ can be derived by choosing the weighting factor ρ in the following range:

$$\rho \ge \sqrt{3\gamma^2/(\gamma^2 - 1)} \tag{19}$$

Therefore, we have proved that the function $U = -\rho^2 V_r V_\theta^2/2r$ with ρ constrained by Eq. (19) is truly a solution of the HJPDI with $\gamma > 1$. The resulting H_∞ robust guidance law can be computed from Eq. (15) as

$$\begin{bmatrix} u_r \\ u_\theta \end{bmatrix} = \begin{bmatrix} \dfrac{\partial U}{\partial V_r} \\[2mm] \dfrac{\partial U}{\partial V_\theta} \end{bmatrix} = \begin{bmatrix} -\dfrac{\rho^2 V_\theta^2}{2r} \\[2mm] -\dfrac{\rho^2 V_r V_\theta}{r} \end{bmatrix} \tag{20}$$

It can be seen that the H_∞ guidance law does not need the information of the target's acceleration w_r and w_θ; nevertheless, the line-of-sight information provided by the seeker is still required.

B. $\gamma = 1$

In this case, the HJPDI degenerates into the following first-order linear partial differential inequality:

$$V_r\left(\frac{\partial U}{\partial r}\right) + \frac{V_\theta^2}{r}\left(\frac{\partial U}{\partial V_r}\right) - \frac{V_r V_\theta}{r}\left(\frac{\partial U}{\partial V_\theta}\right) + \frac{\rho^2 V_\theta^4}{2r^2} \le 0 \tag{21}$$

A possible solution can be identified from the following set:

$$V_r\left(\frac{\partial U}{\partial r}\right) + \frac{V_\theta^2}{r}\left(\frac{\partial U}{\partial V_r}\right) \le 0 \tag{22a}$$

$$-\frac{V_r V_\theta}{r}\left(\frac{\partial U}{\partial V_\theta}\right) + \frac{\rho^2 V_\theta^4}{2r^2} = 0 \tag{22b}$$

Solving Eq. (22b) for U gives

$$U = c + \frac{\rho^2 V_\theta^4}{8r V_r} \tag{23}$$

where c is a positive constant to ensure that U is positive. By using this U in Eq. (22a), we can rewrite Eq. (22a) as

$$-\frac{\rho^2 V_\theta^4}{8r^2} - \frac{\rho^2 V_\theta^6}{8r^2 V_r^2} \le 0 \tag{24}$$

which is satisfied automatically. Hence, the U given in Eq. (23) is a solution of the HJPDI with $\gamma = 1$. The corresponding missile acceleration command turns out to be

$$u_r = \frac{\partial U}{\partial V_r} = -\frac{\rho^2 V_\theta^4}{8r V_r^2} \tag{25a}$$

$$u_\theta = \frac{\partial U}{\partial V_\theta} = -\frac{\rho^2 V_\theta^3}{2r V_r} \tag{25b}$$

C. $\gamma < 1$

A smaller value of the disturbance attenuation level γ means that the interceptive performance is less sensitive to the target's maneuvering; however, more stringent conditions need to be imposed on the corresponding robust guidance law. Inspecting the HJPDI (14), we search for the possible solution candidate from the following auxiliary inequality:

$$\frac{1}{2}\left(\frac{1}{\gamma^2} - 1\right)\left(\frac{\partial U}{\partial V_r}\right)^2 + \frac{V_\theta^2}{r}\left(\frac{\partial U}{\partial V_r}\right) + \frac{\rho^2 V_\theta^4}{2r^2} \le 0 \tag{26}$$

By treating Eq. (26) as a second-order algebraic inequality in $\partial U/\partial V_r$, the solution set can be found as

$$\frac{-1 - \sqrt{1 - \rho^2(1/\gamma^2 - 1)}}{1/\gamma^2 - 1}\frac{V_\theta^2}{r} \le \frac{\partial U}{\partial V_r} \le \frac{-1 + \sqrt{1 - \rho^2(1/\gamma^2 - 1)}}{1/\gamma^2 - 1}\frac{V_\theta^2}{r} \tag{27}$$

Therefore, $\partial U/\partial V_r$ can be expressed as

$$\frac{\partial U}{\partial V_r} = \lambda\frac{V_\theta^2}{r} \tag{28}$$

where

$$\frac{-1 - \sqrt{1 - \rho^2(1/\gamma^2 - 1)}}{1/\gamma^2 - 1} \le \lambda \le \frac{-1 + \sqrt{1 - \rho^2(1/\gamma^2 - 1)}}{1/\gamma^2 - 1} \tag{29}$$

It can be seen that in this case the weighting factor ρ cannot be assigned arbitrarily and must be within the interval

$$0 \le \rho \le \frac{1}{\sqrt{1/\gamma^2 - 1}} \tag{30}$$

The integration of Eq. (28) gives

$$U = \lambda\frac{V_r V_\theta^2}{r} \tag{31}$$

This U is positive by noting that λ is negative from Eq. (29), and the closing speed V_r also must be negative to ensure successful interception. We substitute this U into Eq. (14) to check whether this form of U can satisfy the HJPDI. After some manipulations, we get

$$\frac{1}{2}\left(\frac{1}{\gamma^2} - 1\right)\frac{\lambda^2 V_\theta^4}{r^2} + \frac{\lambda V_\theta^4}{r^2} + \frac{\rho^2 V_\theta^4}{2r^2} + \left(\frac{1}{\gamma^2} - 1\right)\frac{2\lambda^2 V_r^2 V_\theta^2}{r^2} - \frac{3\lambda V_r^2 V_\theta^2}{r^2} \le 0 \tag{32}$$

Define the total velocity $V(t)$ as $V^2(t) = V_r^2(t) + V_\theta^2(t)$, and recast the preceding equation into the following form:

$$\frac{1}{2}\left(\frac{1}{\gamma^2} - 1\right)\frac{\lambda^2 V_\theta^2}{r^2}\left(V^2 - V_r^2\right) + \frac{\lambda V_\theta^2}{r^2}\left(V^2 - V_r^2\right) + \frac{\rho^2 V_\theta^4}{2r^2} + \left(\frac{1}{\gamma^2} - 1\right)\frac{2\lambda^2 V_\theta^2\left(V^2 - V_\theta^2\right)}{r^2} - \frac{3\lambda V_r^2 V_\theta^2}{r^2} \le 0 \tag{33}$$

The combination of the similar terms yields

$$\left[\frac{5}{2}\left(\frac{1}{\gamma^2}-1\right)\lambda^2+\lambda\right]\frac{V_\theta^2 V^2}{r^2}+\left[\frac{-1}{2}\left(\frac{1}{\gamma^2}-1\right)\lambda^2-4\lambda\right]$$

$$\times\frac{V_r^2 V_\theta^2}{r^2}+\left[-2\left(\frac{1}{\gamma^2}-1\right)\lambda^2+\frac{1}{2}\rho^2\right]\frac{V_\theta^4}{r^2}\le 0 \tag{34}$$

A sufficient condition for Eq. (34) to be satisfied is that each coefficient of the three terms in Eq. (34) is negative; i.e.,

$$\tfrac{5}{2}(1/\gamma^2-1)\lambda^2+\lambda\le 0 \tag{35a}$$

$$\tfrac{-1}{2}(1/\gamma^2-1)\lambda^2-4\lambda\le 0 \tag{35b}$$

$$-2(1/\gamma^2-1)\lambda^2+\tfrac{1}{2}\rho^2\le 0 \tag{35c}$$

The solution set of each inequality becomes

$$\frac{\frac{-2}{5}}{\gamma^{-2}-1}\le\lambda\le 0 \tag{36a}$$

$$\frac{-8}{\gamma^{-2}-1}\le\lambda\le 0 \tag{36b}$$

$$\lambda\ge\frac{\rho}{2\sqrt{\gamma^{-2}-1}}\qquad\text{or}\qquad\lambda\le\frac{-\rho}{2\sqrt{\gamma^{-2}-1}} \tag{36c}$$

The final admissible range of λ is determined by the intersection of the intervals in Eqs. (29) and (36). The answer depends on the magnitude of the weighting factor ρ for which the interval in Eq. (30) is divided into two portions: 1) $0\le\rho\le[\frac{4}{5}/(\sqrt{1/\gamma^2-1})]$: The allowable interval of λ is given by

$$\frac{\frac{-2}{5}}{1/\gamma^2-1}\le\lambda\le\frac{-\rho}{2\sqrt{1/\gamma^2-1}} \tag{37}$$

2) $[\frac{4}{5}/(\sqrt{1/\gamma^2-1})]<\rho\le[1/(\sqrt{1/\gamma^2-1})]$: There is no intersection among the intervals of λ.

In summary, the solution of the HJPDI with $\gamma<1$ is given by Eq. (31) where the constant λ locates within the interval of Eq. (37). The associated robust guidance law is given by

$$\begin{bmatrix}u_r\\u_\theta\end{bmatrix}=\begin{bmatrix}\dfrac{\partial U}{\partial V_r}\\[2mm]\dfrac{\partial U}{\partial V_\theta}\end{bmatrix}=\begin{bmatrix}\dfrac{\lambda V_\theta^2}{r}\\[2mm]\dfrac{2\lambda V_r V_\theta}{r}\end{bmatrix} \tag{38}$$

The constant λ can be regarded as the missile's navigation gain whose value depends on the disturbance attenuation level γ to be achieved and the weighting factor ρ. The smaller the γ is, the more robust the missile will be with respect to the variations of the target's maneuvers. Although γ can be assigned arbitrarily near zero, it can be seen from Eq. (37) that the admissible interval of the navigation gain λ shrinks very rapidly as γ approaches zero. The variation of λ with respect to ρ and γ is depicted in Fig. 2.

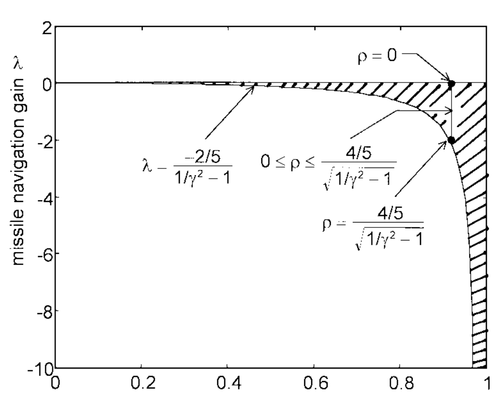

Fig. 2 Relation of λ and γ for various ρ.

V. Performance Evaluation of the H_∞ Guidance Law

The performance of the H_∞ robust guidance law is measured by Eq. (12), which says that the disturbance (i.e., the target's maneuvers) attenuation level can be kept less than γ for an arbitrary target acceleration $\boldsymbol{w}\in\boldsymbol{L_2}$ (i.e., for such targets whose

$$\int_0^\infty \boldsymbol{w}^T\boldsymbol{w}\,\mathrm{d}t$$

is finite). However, because there are many possibilities of the allowable target's accelerations, it is not possible to identify the actual trajectories of the missile in advance. Nevertheless, some underlying properties of the missile's trajectories can still be captured by the analytical solutions with respect to nonmaneuvering targets. Trajectory characteristics, including capture area, cumulative velocity increment, and miss distance, will be discussed analytically in this section. As to maneuvering targets, the actual trajectories of the missile can only be computed numerically, as considered in the next section.

Let $w_r=w_\theta=0$ and substitute the missile acceleration components u_r and u_θ derived from Eq. (20) into Eq. (9) to yield the governing equations as

$$\dot r=V_r,\qquad r(0)=r_0 \tag{39a}$$

$$\dot V_r=(\rho^2/2+1)\left(V_\theta^2/r\right),\qquad V_r(0)=V_{r_0} \tag{39b}$$

$$\dot V_\theta=(\rho^2-1)(V_rV_\theta/r),\qquad V_\theta(0)=V_{\theta_0} \tag{39c}$$

Examining the preceding equations, we know that $V_r(t)$ is a monotonically increasing function of time t along the whole trajectory. To ensure successful interception, the closing speed $\dot r=V_r$ must be negative; hence $V_r(t)$ must be increasing from $V_{r_0}<0$ to at most $V_r(t_f)=0$ at the pursuit end. Consequently, it may happen that $r(t_f)\ne 0$ when $V_r(t_f)=0$, which leads to nonzero miss distance. On the other hand, the sign of $\dot V_\theta(t)$ depends on the magnitude of ρ: 1) $\rho>1$, 2) $\rho=1$, and 3) $\rho<1$. These three cases will be discussed separately.

1) $\rho>1$: Dividing Eq. (39c) by Eq. (39b), we get

$$\frac{\mathrm{d}V_r}{\mathrm{d}V_\theta}=\frac{\rho^2/2+1}{\rho^2-1}\frac{V_\theta}{V_r} \tag{40}$$

Direct integration gives

$$V_r^2-\frac{\rho^2/2+1}{\rho^2-1}V_\theta^2=V_{r_0}^2-\frac{\rho^2/2+1}{\rho^2-1}V_{\theta_0}^2=\text{const} \tag{41}$$

The relation between r and V_θ can be established from Eq. (39c),

$$\frac{\mathrm{d}V_\theta}{V_\theta}=(\rho^2-1)\frac{\mathrm{d}r}{r} \tag{42}$$

Integrating both sides leads to

$$V_\theta=V_{\theta_0}(r/r_0)^{\rho^2-1} \tag{43}$$

Substituting Eq. (43) into Eq. (41) and rearranging the result, we get

$$V_r^2-V_{r_0}^2=\frac{\rho^2/2+1}{\rho^2-1}V_{\theta_0}^2\left[\left(\frac{r}{r_0}\right)^{2\rho^2-2}-1\right] \tag{44}$$

Now we can identify the conditions under which a zero miss distance is achievable. Evaluating Eq. (44) at the pursuit and setting r to zero, we have

$$V_{r_f}^2=V_{r_0}^2-\frac{\rho^2/2+1}{\rho^2-1}V_{\theta_0}^2\ge 0 \tag{45}$$

Therefore, the initial condition (V_{r_0},V_{θ_0}) needs to satisfy the following constraint:

$$\left|\frac{V_{r_0}}{V_{\theta_0}}\right|\ge\sqrt{\frac{\rho^2/2+1}{\rho^2-1}} \tag{46}$$

It determines the geometric relation when a missile can intercept a target with a zero miss distance. Inequality (46) characterizes the capture area for the robust guidance law when pursuing nonmaneuvering targets. The capture area is enlarging with increasing ρ. In the limiting case as ρ approaches ∞, the capture area becomes

$$\left|V_{r_0}\big/V_{\theta_0}\right| \geq 1/\sqrt{2} \tag{47}$$

But if ρ approaches one, the capture area is shrunk to zero. In other words, the missile cannot intercept the target with a zero miss distance in this case. This point will be further explained in the solution with $\rho = 1$.

When the initial condition (V_{r_0}, V_{θ_0}) does not fall within the capture area, the miss distance can be determined by letting $V_r = \dot{r} = 0$ in Eq. (44) and finding the minimum distance between target and missile (the definition of miss distance) as

$$r_{\min} = r_0\left[1 - \frac{\rho^2 - 1}{\rho^2/2 + 1}\left(\frac{V_{r_0}}{V_{\theta_0}}\right)^2\right]^{1/(2\rho^2-2)} \tag{48}$$

Having obtained the relation of V_r and V_θ, we are ready to evaluate the final time of the interception for this robust guidance law. The closing speed V_r is calculated first. From Eq. (44), we have

$$\dot{r} = -\sqrt{\frac{\rho^2/2 + 1}{\rho^2 - 1}V_{\theta_0}^2\left(\frac{r}{r_0}\right)^{2\rho^2-2} + K} \tag{49}$$

where $K = V_{r_0}^2 - V_{\theta_0}^2(\rho^2/2+1)/(\rho^2-1)$. Integrating the preceding equation, we obtain

$$t = \int_r^{r_0}\left(\mathrm{d}r\bigg/\sqrt{\frac{\rho^2/2+1}{\rho^2-1}V_{\theta_0}^2\left(\frac{r}{r_0}\right)^{2\rho^2-2} + K}\right) \tag{50}$$

The interception time t_f can be obtained by setting $r = 0$ in the lower limit of the preceding integration when the initial engagement condition falls within the capture area of Eq. (46); otherwise, the lower limit is replaced by $r = r_{\min}$. Finally, we compute the cumulative velocity increment ΔV as

$$\Delta V = \int_0^{t_f} |u|\,\mathrm{d}t = \int_0^{t_f}\sqrt{u_r^2 + u_\theta^2}\,\mathrm{d}t$$

$$= \rho^2\int_0^{t_f}\frac{V_\theta}{r}\sqrt{V_\theta^2\big/4 + V_r^2}\,\mathrm{d}t \tag{51}$$

Changing the independent variable from t to r via Eq. (49) and using V_r and V_θ from Eqs. (44) and (43), we can express ΔV in the concise form of elliptic integration

$$\frac{\Delta V}{|V_{\theta_0}|} = \frac{\rho^2}{\rho^2 - 1}\int_{y_l}^{1}\sqrt{\frac{A_1 y^2 + 1}{B_1 y^2 + 1}}\,\mathrm{d}y \tag{52}$$

where

$$y = \left(\frac{r}{r_0}\right)^{\rho^2-1}, \qquad A_1 = \frac{3(\rho^2+1)}{4(\rho^2-1)}\left(\frac{V_{\theta_0}}{K}\right)^2$$

$$B_1 = \frac{\rho^2/2+1}{\rho^2-1}\left(\frac{V_{\theta_0}}{K}\right)^2$$

The lower limit of integration y_l can be set to 0 or $(r_{\min}/r_0)^{\rho^2-1}$, depending on the initial conditions.

2) $\rho = 1$: From Eq. (39c), we have $V_\theta(t) = V_{\theta_0} = \text{const}$. Applying this result to Eq. (39b) yields

$$r\ddot{r} = \tfrac{3}{2}V_{\theta_0}^2 \tag{53}$$

Noting that $\ddot{r} = \mathrm{d}\dot{r}/\mathrm{d}t = (\mathrm{d}\dot{r}/\mathrm{d}r)(\mathrm{d}r/\mathrm{d}t)$, we can rewrite Eq. (53) as

$$\dot{r}\,\mathrm{d}\dot{r} = \frac{3V_{\theta_0}^2}{2}\frac{\mathrm{d}r}{r} \tag{54}$$

The integration gives

$$\dot{r} = -\sqrt{3V_{\theta_0}^2\ln(r/r_0) + V_{r_0}^2} \tag{55}$$

As mentioned earlier, the zero-miss-distance interception cannot be achieved in the case of $\rho = 1$ because when we set $r = 0$ in Eq. (55), there is no corresponding solution for $\dot{r}$. The minimum distance $r_{\min}$ between target and missile is determined from Eq. (55) by setting $\dot{r} = 0$:

$$r_{\min}|_{\rho=1} = r_0\exp\left(\frac{-V_{r_0}^2}{3V_{\theta_0}^2}\right) \tag{56}$$

The relation between r and time t can be found by the integration of Eq. (55), leading to

$$t = \int_r^{r_0}\frac{\mathrm{d}r}{\sqrt{3V_{\theta_0}^2\ln(r/r_0) + V_{r_0}^2}} \tag{57}$$

Finally, the cumulative velocity increment is obtained by substituting V_r from Eq. (55) and $V_\theta(t) = V_{\theta_0}$ into Eq. (51):

$$\frac{\Delta V}{|V_{\theta_0}|} = \int_0^{B_2}\sqrt{\frac{A_2 - y}{B_2 - y}}\,\mathrm{d}y \tag{58}$$

where

$$y = \ln\frac{r_0}{r}, \qquad A_2 = \frac{1}{12} + \frac{V_{r_0}^2}{3V_{\theta_0}^2}, \qquad B_2 = \frac{V_{r_0}^2}{3V_{\theta_0}^2}$$

3) $\rho < 1$: The formulas derived for the case of $\rho > 1$ are still valid for $\rho < 1$. The only difference is that the capture area does not exist in this case as can be seen from Eq. (44) where when r is set to zero, there is no solution for V_r. The miss distance $r_{\min}$ is the same as Eq. (48) with ρ constrained by $0 \leq \rho < 1$. It can be seen that $r_{\min}$ is increasing as ρ is decreasing to 0. The largest miss distance occurs at $\rho = 0$, which can be found as

$$r_{\min}|_{\rho=0} = \frac{r_0}{\sqrt{1 + \left(V_{r_0}\big/V_{\theta_0}\right)^2}} \tag{59}$$

Referring to Eq. (10), we see that ρ is a tradeoff between interception performance and required missile acceleration, and smaller ρ implies smaller weight on interception performance, which in turn implies that the performance becomes worse. Hence, it is natural to see that $r_{\min}|_{\rho=0} > r_{\min}|_{\rho=1}$ as can be verified by comparing Eq. (59) with Eq. (56).

VI. Numerical Results

In this section, we exploit numerical simulations to justify the feasibility of the preceding theoretical derivations. For clarity, we will divide the simulations into two parts. One concerns nonmaneuvering target interception, and the other concerns maneuvering target interception. To make the analysis independent of the physical units, we normalize the governing equations to their dimensionless forms in terms of the following dimensionless variables. The relative distance r is normalized to $\bar{r} = r/r_0$; the relative velocities V_r and V_θ are normalized to $\bar{V}_r = V_r/V_0$ and $\bar{V}_\theta = V_\theta/V_0$, respectively, where $V_0 = \sqrt{(V_{r_0}^2 + V_{\theta_0}^2)} = \sqrt{[\dot{r}_0^2 + (r_0\dot{\theta}_0)^2]}$; the acceleration components u_r and u_θ are normalized to $\bar{u}_r = u_r/(V_0^2/r_0)$ and $\bar{u}_\theta = u_\theta/(V_0^2/r_0)$, respectively; and the time t is normalized to $\tau = t/(r_0/V_0)$. In the following, the variables with the overline symbol denote dimensionless variables.

The performance and robustness of four missile guidance laws will be tested and compared. Their acceleration commands are listed next.

1) H_∞ guidance law with $\gamma > 1$: $\bar{u}_r = -\rho^2\bar{V}_\theta^2/(2\bar{r})$, and $\bar{u}_\theta = -\rho^2\bar{V}_r\bar{V}_\theta/\bar{r}$.

2) H_∞ guidance law with $\gamma < 1$: $\bar{u}_r = \lambda\bar{V}_\theta^2/\bar{r}$, and $\bar{u}_\theta = 2\lambda\bar{V}_r\bar{V}_\theta/\bar{r}$, where λ is constrained by Eq. (37).

3) Pure proportional navigation[16] (PPN) guidance law: $\bar{u}_r = -\lambda\bar{V}_M(\mathrm{d}\theta/\mathrm{d}\tau)\sin(k\theta + \alpha_0)$, and $\bar{u}_\theta = \lambda\bar{V}_M(\mathrm{d}\theta/\mathrm{d}\tau)\cos(k\theta + \alpha_0)$,

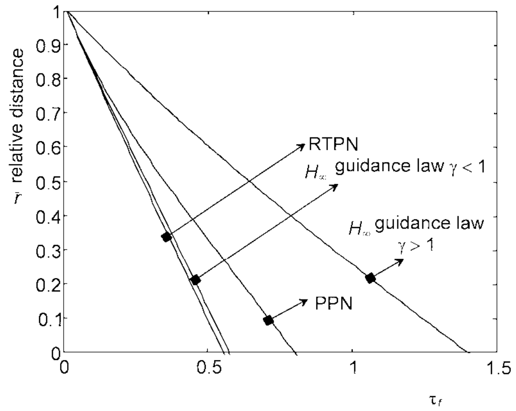

Fig. 3 Trajectories for different guidance laws.

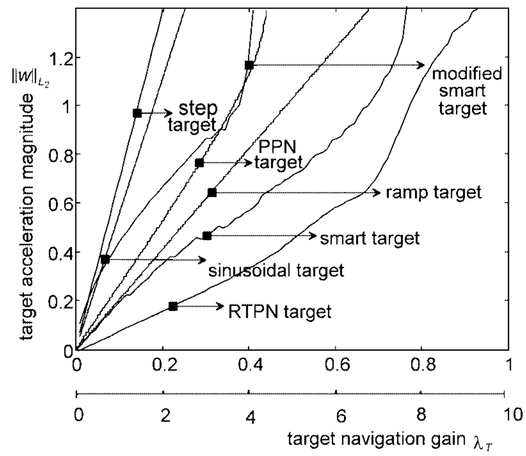

Fig. 4 Relation between target navigation gain and target acceleration magnitude.

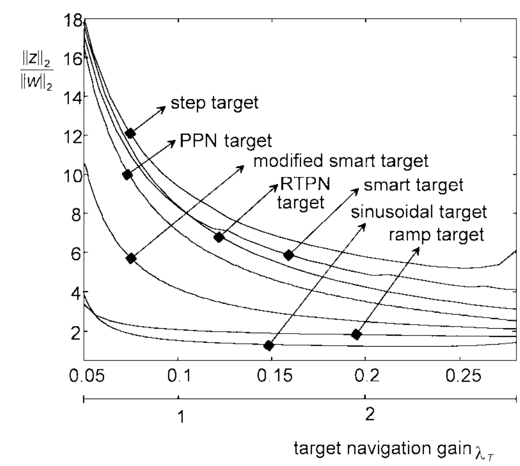

Fig. 5 Robustness of L_2 gain for H_∞ guidance law with $\gamma > 1$.

where $\bar{V}_M$ is the normalized missile's velocity, $k = \lambda - 1$ with λ being the navigation gain, and α_0 is the initial aspect angle of $\boldsymbol{V}_M$ with respect to the inertial reference line (see Fig. 1).

4) Realistic true proportional navigation[16] (RTPN): $\bar{u}_r = 0$, and $\bar{u}_\theta = -\lambda(\mathrm{d}\theta/\mathrm{d}\tau)\bar{V}_r$.

A. Nonmaneuvering Targets

Figure 3 depicts the four missile trajectories for pursuing nonmaneuvering targets. From Fig. 3, it can be seen that the H_∞ robust guidance law with $\gamma > 1$ spends the longest time to intercept the target, whereas the H_∞ robust guidance law with $\gamma < 1$ requires much less interceptive time. Besides the interceptive time, we need to consider the energy expenditure. From the computation of the cumulative velocity increment ΔV, we find that the H_∞ robust guidance law with $\gamma < 1$ ($\Delta V = 1.4426$) spends almost equivalent energy with PPN ($\Delta V = 1.4862$), both requiring less energy consumption. Actually, it is recognized that the conventional PN schemes provide excellent interceptive strategy for nonmaneuvering targets, whereas the robust guidance law, especially with $\gamma > 1$, exhibits awkward performance for nonmaneuvering targets. This result is not surprising because the robust guidance law is designed under the assumption that the target's motion is unknown or uncertain. If the target's motion is completely known, the performance of the robust guidance law may become too conservative, and the best guidance law can be otherwise obtained via the optimization process with respect to the given target's trajectory.

B. Maneuvering Targets

In this part, we employ seven maneuvering strategies for targets to test the robust capability of the H_∞ guidance laws. These seven different targets, which are widely discussed in the literature, include the following.

1) Smart target[19]: $\bar{w}_r = 0$, and $\bar{w}_\theta = \lambda_T/(\bar{r}\,\mathrm{d}\theta/\mathrm{d}\tau)$.

2) Modified smart target[20]: $\bar{w}_r = 0$, and $\bar{w}_\theta = \lambda_T/(\mathrm{d}\bar{r}/\mathrm{d}\tau \cdot \mathrm{d}\theta/\mathrm{d}\tau)$.

3) Sinusoidal target[21]: $\bar{w}_r = \bar{w}_\theta = \lambda_T \sin(\tau\,\mathrm{d}\theta/\mathrm{d}\tau)$.

4) Ramp target[21]: $\bar{w}_r = \bar{w}_\theta = \lambda_T\tau$.

5) Step target[21]: $\bar{w}_r = 0$, and $\bar{w}_\theta = \lambda_T$.

6) RTPN target[16]: $\bar{w}_r = 0$, and $\bar{w}_\theta = -\lambda_T(\mathrm{d}\theta/\mathrm{d}\tau)\bar{V}_r$.

7) PPN target[16]: $\bar{w}_r = -\bar{V}_T(\mathrm{d}\theta/\mathrm{d}\tau)\sin(k\theta + \beta_0)$, and $\bar{w}_\theta = \bar{V}_T(\mathrm{d}\theta/\mathrm{d}\tau)\sin(k\theta + \beta_0)$, where $\bar{V}_T$ is the normalized target's velocity, $k = \lambda_T - 1$ with λ_T being the target's navigation gain, and β_0 is the initial aspect angle of $\boldsymbol{V}_T$ with respect to an inertial reference line.

It should be emphasized that these target models are only for demonstration purpose, and other more practical target models can be chosen to test the H_∞ guidance law. Whatever target model is used, the H_∞ guidance law can guarantee the disturbance attenuation level lower than γ as long as target's acceleration is finite.

Here, we choose the target's navigation gains λ_T and the initial engagement conditions as the uncertainty disturbances to test the robustness properties of the H_∞ guidance laws. From the definition of the system's L_2 gain [see Eq. (12)], we recognize that when γ is small, it indicates that good interceptive performance can be preserved in the presence of large target maneuvers (i.e., large $\|w\|_2$). In the following, the simulations will show that by using the H_∞ robust guidance law the system's L_2 gain can be kept below γ under variations of the target's maneuvering strategy, the target's navigation gains, and the initial engagement conditions.

1. *Robustness with Respect to Varying Target Maneuvering*

In Fig. 4, we show the variations of the target's maneuvering energy with respect to the target's navigation gain under the action of the H_∞ robust guidance law with $\gamma > 1$. There are two abscissas in Fig. 4, where the one with a scale larger than 1 is fitted for step target, ramp target, sinusoidal target, RTPN target, and PPN target; the other one with a scale less than 1 is fitted to smart target and modified smart target. This figure helps us to select the reasonable range of each target's navigation gain. For example, for the RTPN target, the gain can be selected as 10, but for the smart target, the gain's reasonable range is between 0.6 and 0.8 as can be seen from Fig. 4. Because in practical implementation the target's maneuvering energy must be finite, if the gain does not fall within the appropriate range, the target's energy will approach infinity. Although the H_∞ robust guidance law can maintain admissible performance for almost arbitrary maneuvering targets, it is restricted to pursue the targets whose acceleration energy belongs to the set of L_2, i.e., $\|w\|_2$ is finite. Indeed, the target with infinite acceleration energy does not exist in the real world. There are many other practical limits on w!

Now we demonstrate the robust property of the H_∞ guidance law with respect to the target's varying navigation gains as selected from Fig. 4 for different missile guidance laws. In Figs. 5 and 6, the two H_∞ guidance laws demonstrate robustness against the seven aforementioned maneuvering targets, being able to keep the L_2 gain smaller than γ for different target navigation gains and for different target maneuvering strategies. A comparison between the results of

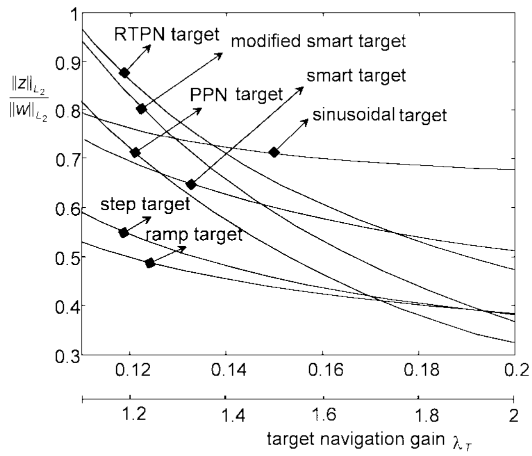

Fig. 6 Robustness of L_2 gain for H_∞ guidance law with γ < 1.

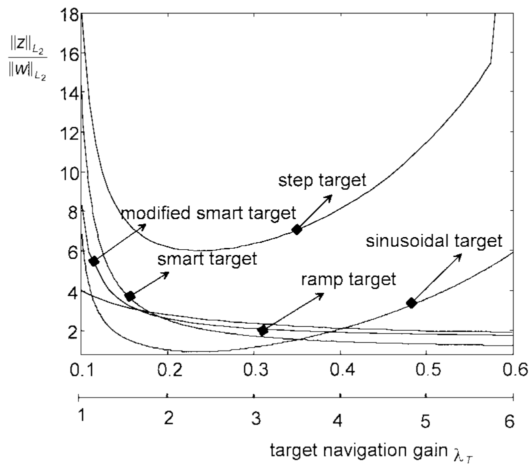

Fig. 7 Robustness of L_2 gain for RTPN guidance law.

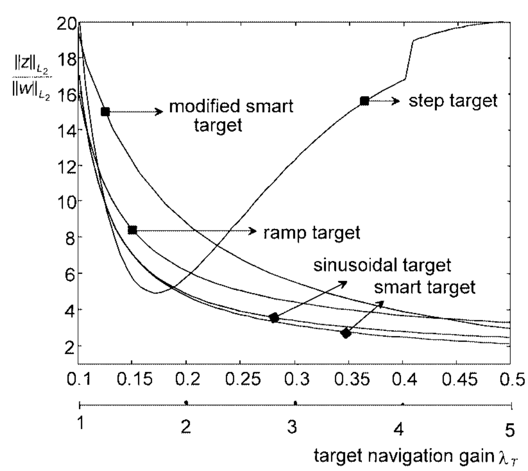

Fig. 8 Robustness of L_2 gain for PPN guidance law.

Figs. 5 and 6 reveals that the robustness of the H_∞ guidance law with $\gamma < 1$ is better than that of the H_∞ guidance law with $\gamma > 1$. On the other hand, if the missile's guidance law employs the conventional proportional navigations such as RTPN and PPN (see Figs. 7 and 8, respectively), the disturbance attenuation level may diverge for some specific maneuvering targets. We can find from Fig. 7 that the RTPN missile guidance law has a poor ability to pursue the step target and the sinusoidal target, whereas from Fig. 8 we see that the PPN missile guidance law has a poor ability to pursue the step target. In summary, among the four missile guidance laws, the H_∞ guidance laws, especially for $\gamma < 1$, exhibit the most robust performance with respect to the variation of the target's navigation gains.

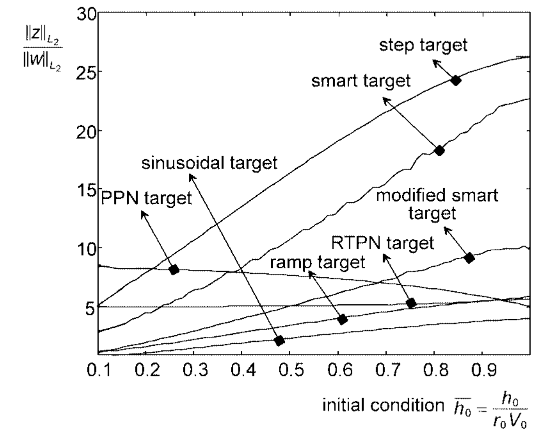

Fig. 9 System L_2 gain vs $\bar{h}_0$ for H_∞ guidance law with γ > 1.

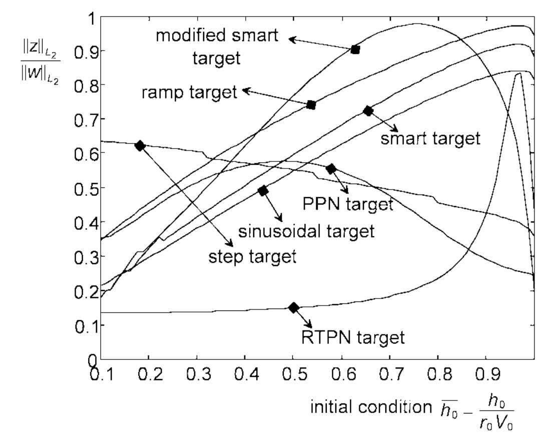

Fig. 10 System L_2 gain vs $\bar{h}_0$ for H_∞ guidance law with γ < 1.

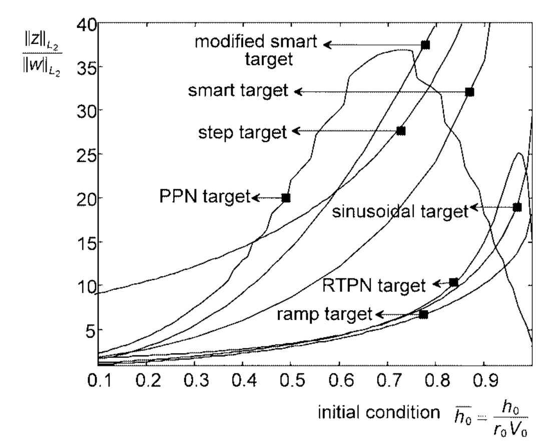

Fig. 11 System L_2 gain vs $\bar{h}_0$ for RTPN guidance law.

2. *Robustness with Respect to Varying Initial Engagement Conditions*

In this part, we wish to investigate the robust ability of H_∞ guidance laws with respect to the variations of the initial engagement conditions. For convenience, we introduce the initial angular momentum $h_0 = r_0^2\dot{\theta}_0$ as an index to reflect the impact of initial conditions. Figures 9–12 express the robustness of the four missile guidance laws with respect to changing h_0 for the seven different target maneuvers. The abscissa in these figures is the normalized initial angular momentum $\bar{h}_0 = h_0/(r_0 V_0)$. The magnitude of $\bar{h}_0$ is between 0 and 1. For a tail-chase initial condition, $\bar{h}_0 = 0$, whereas for the worst case where the target is escaping fully tangentially, $\bar{h}_0 = 1$. Hence, when we increase $\bar{h}_0$ from 0 to 1, we impose increasingly stringent engagement conditions on the missile.

It is observed that for any $\bar{h}_0$ between 0 and 1, the two kinds of H_∞ guidance laws can maintain an excellent disturbance attenuation

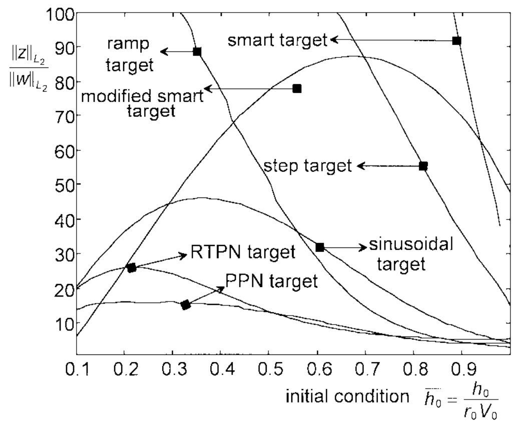

Fig. 12 System L_2 gain vs $\bar{h}_0$ for PPN guidance law.

ability, whereas the performances of RTPN and PPN fluctuate dramatically with changing $\bar{h}_0$. As expected, the robustness against an initial condition variation for the H_∞ guidance law with $\gamma < 1$ is better than that with $\gamma > 1$, being able to maintain the L_2 gain below 1. It is emphasized that the ability of maintaining the L_2 gain below 1 for the H_∞ guidance law with $\gamma < 1$ is not only valid for the seven targets considered here but also valid for all the targets with finite accelerations. The H_∞ guidance law with $\gamma > 1$ only can maintain the L_2 gain below some value about 25. On the other hand, the performances of RTPN and PPN guidance laws are divergent for some specific targets and under some specific range of $\bar{h}_0$ (see Figs. 11 and 12).

C. Remarks

1) From the simulation results we can find that the performance of the H_∞ guidance law is rather conservative. This conservativeness seems to be unavoidable because the H_∞ performance (i.e., L_2 gain $\leq \gamma$) must be preserved for any target with bounded acceleration. The more target information we know, the less conservative guidance law we can design. Hence, if information about the target's acceleration is accessible, the specific guidance law designed according to this target information is undoubtedly superior to the H_∞ guidance law. The conservativeness of the H_∞ guidance law can only be justified under the circumstances where the target's acceleration is unpredictable. This is the main reason that we make the unpredictability assumption of the target's acceleration in the present paper. This unpredictability assumption is not restricted to the guidance law design but is common for all H_∞ design problems.

2) One of the ways to reduce the conservativeness of the H_∞ guidance law is via the proper selection of the HJPDI solution. Because the solution of HJPDI is not unique, it is possible to choose a specific solution that provides the most effective interception with respect to a given kind of target.

3) The extension of the present result to the three-dimensional real world is not hard. In the three-dimensional case, the equation of motion is described in the spherical coordinates[15] (r, θ, ϕ) as a set of three second-order differential equations that can be recast into the form of Eq. (4) where the new state is defined as $\boldsymbol{x} = [r \ \ \phi \ \ V_r \ \ V_\theta \ \ V_\phi]^T$. Using Eq. (4), we can derive the associated HJPDI for the three-dimensional guidance problem, and the main task is to find a positive solution for this three-dimensional HJPDI.

VII. Conclusions

The nonlinear H_∞ control theory has been exploited in this paper to design a guidance law for homing missiles. By regarding target maneuvers as disturbances inputs, we reformulate the missile guidance problem as a nonlinear disturbance attenuation control problem. After solving the associated Hamilton–Jacobi partial differential inequality, we obtain three kinds of H_∞ guidance laws that possess excellent performance robustness against maneuvering targets and against variations of initial engagement conditions. As compared with the proportional navigation schemes, the performance of the proposed H_∞ robust guidance laws is shown to be more insensitive to variations of target's maneuvers.

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